Conformal Field Theory and Gravity

Solutions to Problem Set 12 Fall 2024

1. Vector fields in AdS

(a) Massive vector fields have $D-1$ degrees of freedom, given that the conjugate momentum to the component A^0 is $F_{00}=0$, hence A^0 is non-dynamical. Massless vectors, have $D-2$ because another one of their components can be removed with a gauge transformation.

(b) Trivial use of Euler-Lagrange's equations.

(c) For the $D-1$ components with $i \neq z$, the equations of motion yield

$$\partial_i F^{ib} + \partial_z F^{zb} + (4-D)z^{-1}F^{zb} = z^{-2}m^2 A^b, \tag{1}$$

which gives

$$\nu(\nu-1)z^{\nu-2}J_i + (3-d)\nu z^{\nu-2}J_i = m^2 z^{\nu-2}J_i. \tag{2}$$

Thus, we have the result

$$\nu(\nu-1) + (3-d)\nu = m^2. \tag{3}$$

Now we undo the conformal transformation:

$$A_i = \eta_{ij}A^j = z^2 g_{ij}A^j, \tag{4}$$

and conclude that A_i has weight 1 under dilation $x^i \mapsto \Omega x^i, z \mapsto \Omega z$, just as we would expect from a free vector field. Hence, the weight of J is

$$\Delta_J = z^{-\nu}A = 1+\nu, \tag{5}$$

and

$$(\Delta-1)(\Delta-2) + (3-d)(\Delta-1) = m^2, \tag{6}$$

or

$$(\Delta-1)(\Delta+1-d) = m^2. \tag{7}$$

(d) For $m^2=0$, we find

$$\Delta = d-1 \quad \text{or} \quad \Delta = 1. \tag{8}$$

Here, $\Delta = d-1$ corresponds to a conserved boundary current J^i, and $\Delta = 1$ corresponds to a boundary potential (background source) A_i. If $A_i = 0$, then it corresponds to a globally conserved current.

(e) For A_z: For $m^2 = 0$, we have gauge symmetry $\delta A_a = \nabla_a \alpha$, so we can impose the gauge condition, e.g., $A_z = 0$, giving $D - 2 = d - 1$ degrees of freedom. Using the z-equation of motion:

$$\partial_i F^{iz} = 0, \quad F^{zz} = 0, \tag{9}$$

we get that J^i is a conserved current on the boundary

$$\partial_z(\partial_i A^i) = 0 \implies \partial_i J^i = \nabla_i J^i = 0. \tag{10}$$

Such conserved currents are vectors in $D - 1$ dimensions, satisfying one constraint. Hence they have $D - 2$ degrees of freedom, which matches bulk gauge fields.

2. The bulk to boundary propagator

(a) With the Ansatz $\phi = e^{i\vec{k}\cdot\vec{x}} f_{\vec{k}}(z)$, and using $\Box\phi = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g} g^{\mu\nu}\partial_\nu \phi)$ in Euclidean Poincaré coordinates, it is straight-forward to obtain

$$z^2 f'' - (d-1) z f' - \vec{k}^2 z^2 f = m^2 f \tag{11}$$

In Mathematica, we find the solutions

$$z^{d/2} J_{\Delta - d/2}(-i|\vec{k}|z) \qquad z^{d/2} Y_{\Delta - d/2}(-i|\vec{k}|z) \tag{12}$$

where J_ν and Y_ν are Bessel functions of the first and second kind. The modified Bessel functions I and K are precisely defined in terms of J and Y by including an i factor in the argument. Thus, an equivalent set of solutions is

$$f^{(1)}_{\vec{k}}(z) = z^{d/2} I_{\Delta - d/2}(|\vec{k}|z) \qquad f^{(2)}_{\vec{k}}(z) = z^{d/2} K_{\Delta - d/2}(|\vec{k}|z) \tag{13}$$

(b) As one can find in Wikipedia, $I_\alpha(x) \sim e^x/\sqrt{2\pi x}$ when $x \to \infty$. Thus, $f^{(1)}$ diverges exponentially in the bulk $z \to \infty$.

As you can check in Mathematica (file provided), the normalization of g is such that

$$g_{\vec{k}}(z) \sim z^{d-\Delta} \qquad (z \to 0) \tag{14}$$

This means that

$$\int d^d k \tilde{\phi}'(\vec{k}) e^{i\vec{k}\cdot\vec{x}} g_{\vec{k}}(z) \to z^{d-\Delta} \int d^d k e^{i\vec{k}\cdot\vec{x}} = z^{d-\Delta} \tilde{\phi}(\vec{x}) \qquad (z \to 0) \tag{15}$$

Thus, this is the correct solution with the given boundary condition :

$$\phi(\vec{x}, z) = \int d^d k \tilde{\phi}'(\vec{k}) e^{i\vec{k}\cdot\vec{x}} g_{\vec{k}}(z) \tag{16}$$

(c) It is easiest to show (equivalently) that

$$g_{\vec{k}}(z) \propto \int d^d y e^{-i\vec{k}\cdot\vec{y}} \left(\frac{z}{z^2 + \vec{y}^2}\right)^\Delta \tag{17}$$

Let us prove it by computing the right-hand-side. Using the integral definition of the gamma function, we write

$$\frac{1}{(z^2+\vec{y}^2)^\Delta} = \frac{1}{\Gamma(\Delta)}\int_0^\infty d\alpha \alpha^{\Delta-1} e^{-\alpha(z^2+\vec{y}^2)} \tag{18}$$

This allows to swap the $\vec{y}$ and α integrals and complete the square :

$$\begin{aligned}\int d^d y e^{-i\vec{k}\cdot\vec{y}}\left(\frac{z}{z^2+\vec{y}^2}\right)^\Delta &= \frac{z^\Delta}{\Gamma(\Delta)}\int_0^\infty d\alpha e^{-\alpha z^2}\alpha^{\Delta-1}\int d^d x e^{-\alpha(\vec{x}-\frac{i}{2\alpha}\vec{k})^2-\frac{1}{4\alpha}\vec{k}^2} \\ &= \frac{z^\Delta}{\Gamma(\Delta)}\pi^{d/2}\int_0^\infty d\alpha e^{-\alpha z^2}\alpha^{\Delta-d/2-1}e^{-\vec{k}^2/(4\alpha)}\end{aligned} \tag{19}$$

Rescaling $\alpha \to \alpha/z^2$ gives

$$\int d^d y e^{-i\vec{k}\cdot\vec{y}}\left(\frac{z}{z^2+\vec{y}^2}\right)^\Delta = \frac{z^\Delta}{\Gamma(\Delta)}\pi^{d/2}z^{-2\Delta+d}\int_0^\infty d\alpha \alpha^{\Delta-d/2-1}e^{-\alpha-\frac{\vec{k}^2 z^2}{4\alpha}} \tag{20}$$

It turns out that this last integral is an integral representation of the modified Bessel function K. More specifically,

$$\int_0^\infty d\alpha \alpha^{\lambda-1}e^{-\alpha-\frac{x^2}{4\alpha}} = 2^{-\lambda+1}|x|^\lambda K_\lambda(|x|) \tag{21}$$

Using this property, and after bringing all factors together, this gives

$$\int d^d y e^{-i\vec{k}\cdot\vec{y}}\left(\frac{z}{z^2+\vec{y}^2}\right)^\Delta = \pi^{d/2}\frac{\Gamma(\Delta-\frac{d}{2})}{\Gamma(\Delta)}g_{\vec{k}}(z) \tag{22}$$

(d) Starting from

$$\phi(\vec{x},z) = \int d^d k \tilde{\phi}'(\vec{k})e^{i\vec{k}\cdot\vec{x}}g_{\vec{k}}(z) \tag{23}$$

We insert our previously derived result

$$g_{\vec{k}}(z) = \frac{1}{\pi^{d/2}}\frac{\Gamma(\Delta)}{\Gamma(\Delta-d/2)}\int d^d y e^{-i\vec{k}\cdot\vec{y}}\left(\frac{z}{z^2+\vec{y}^2}\right)^\Delta \tag{24}$$

obtaining

$$\phi(\vec{x},z) = \frac{1}{\pi^{d/2}}\frac{\Gamma(\Delta)}{\Gamma(\Delta-d/2)}\int d^d y d^d k e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\tilde{\phi}'(\vec{k})\left(\frac{z}{z^2+\vec{y}^2}\right)^\Delta \tag{25}$$

which doing some shifts and sign reversals reads

$$\phi(\vec{x},z) = \frac{1}{\pi^{d/2}}\frac{\Gamma(\Delta)}{\Gamma(\Delta-d/2)}\int d^d y d^d k e^{i\vec{k}\cdot\vec{y}}\tilde{\phi}'(\vec{k})\left(\frac{z}{z^2+(\vec{x}-\vec{y})^2}\right)^\Delta \tag{26}$$

We recognize the inverse Fourier transform which gives back $\tilde{\phi}$. Thus,

$$\phi(\vec{x},z) = \int d^d y K(\vec{x},z;\vec{y})\tilde{\phi}(y) \tag{27}$$

where

$$K_\Delta(\vec{x},z;\vec{y}) \equiv \frac{1}{\pi^{d/2}}\frac{\Gamma(\Delta)}{\Gamma(\Delta-d/2)}\left(\frac{z}{z^2+(\vec{x}-\vec{y})^2}\right)^\Delta . \tag{28}$$

3. Correlation functions

(a) Starting from

$$S \sim \frac{1}{G} \int d^{10}x \sqrt{-g} (R + c_1 \alpha' R^2 + c_2 \alpha' R_{\mu\nu} R^{\mu\nu} + \dots) \tag{29}$$

we rescale the metric by making the AdS scale R_{AdS} appear, $g_{\mu\nu} = R_{\text{AdS}}^2 \tilde{g}_{\mu\nu}$. This gives

$$R = R_{\text{AdS}}^{-2} \tilde{R} \qquad R_{\mu\nu} R^{\mu\nu} = R_{\text{AdS}}^{-4} \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} \qquad \sqrt{-g} = R_{\text{AdS}}^{10} \sqrt{-\tilde{g}} \tag{30}$$

Altogether,

$$S \sim \frac{R_{\text{AdS}}^8}{G} \int d^{10}x \sqrt{-\tilde{g}} (\tilde{R} + c_1 \alpha' R_{\text{AdS}}^{-2} \tilde{R}^2 + c_2 \alpha' R_{\text{AdS}}^{-2} \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} + \dots) \tag{31}$$

Let us call this prefactor

$$\frac{1}{\tilde{G}} \equiv \frac{R_{\text{AdS}}^8}{G} = \frac{R_{\text{AdS}}^8}{\ell_p^8} \sim N^2 \tag{32}$$

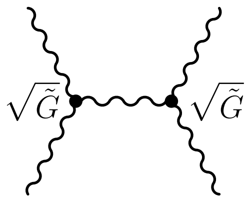
To compute interactions on top of $\tilde{g}_{\mu\nu}^{(\text{AdS})}$, we expand

$$\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}^{(\text{AdS})} + \sqrt{\tilde{G}} h_{\mu\nu} \tag{33}$$

This gives typically

$$S \sim \int d^{10}x \, \underbrace{(\partial h)^2 + \sqrt{\tilde{G}} h \partial h \partial h + \dots}_{\text{Einstein-Hilbert}} + \underbrace{\alpha' R_{\text{AdS}}^{-2} (\partial^2 h \partial^2 h + \sqrt{\tilde{G}} h \partial^2 h \partial^2 h + \dots) + \dots}_{\text{Higher curvature corrections}} \tag{34}$$

The corrections in $\sqrt{\tilde{G}}$ contribute to diagrams such as



$$\sim \tilde{G} \sim \frac{1}{N^2} \tag{35}$$

(and similarily for UV-divergent loop diagrams). Thus, $1/N^2$ corrections correspond to the regime where we need a UV completion to Einstein-Hilbert gravity to be able to make predictions from loop diagrams. These corrections correspond to *quantum gravity* corrections.

On the other hand, $\alpha' R_{\text{AdS}}^{-2}$ controls higher curvature corrections, where the Einstein-Hilbert action itself is not sufficient. In terms of string theory parameters,

$$\alpha' R_{\text{AdS}}^{-2} \sim \ell_s^2 (g_s \ell_s^4 N)^{-1/2} \sim (g_s N)^{-1/2} \sim \lambda^{-1/2} \tag{36}$$

Thus, Einstein-Hilbert is sufficient when $\lambda \to \infty$, and $1/\lambda$ corrections correspond to higher curvature, *stringy*, corrections.

(b) We will write formulas in generic d, where $d = 4$ is the case of interest. We want to compute

$$\langle \mathcal{O}_1(\vec{x}_1)\mathcal{O}_2(\vec{x}_2)\mathcal{O}_3(\vec{x}_3)\rangle = -\frac{\lambda}{\pi^6}\left(\prod_i \frac{\Gamma(\Delta_i)}{\Gamma(\Delta_i - 2)}\right) I \tag{37}$$

where the integral I is

$$\begin{aligned} I &= \int dz d^d x \sqrt{g} \frac{z^{\Delta_1} z^{\Delta_2} z^{\Delta_3}}{(z^2 + (\vec{x}-\vec{x}_1)^2)^{\Delta_1}(z^2 + (\vec{x}-\vec{x}_2)^2)^{\Delta_2}(z^2 + (\vec{x}-\vec{x}_3)^2)^{\Delta_3}} \\ &= \int dz d^d x \frac{1}{z^{d+1}} \frac{1}{(z + (\vec{x}-\vec{x}_1)^2/z)^{\Delta_1}(z + (\vec{x}-\vec{x}_2)^2/z)^{\Delta_2}(z + (\vec{x}-\vec{x}_3)^2/z)^{\Delta_3}} \end{aligned} \tag{38}$$

Let us write the integrand as

$$\int d^d x \prod_{i=1}^{3} \frac{1}{(z + (\vec{x}-\vec{x}_i)^2/z)^{\Delta_i}} = \prod_i \frac{1}{\Gamma(\Delta_i)} \int_0^\infty ds_1 ds_2 ds_3 \prod_i s_i^{\Delta_i - 1} e^{-\sum_i s_i (z + (\vec{x}-\vec{x}_i)^2/z)} \tag{39}$$

We then complete the square in $\vec{x}$, namely

$$\begin{aligned} \sum_i s_i (z + (\vec{x}-\vec{x}_i)^2/z) = \frac{1}{z}(\sum_i s_i)\left(\vec{x} - \frac{\sum_j s_j \vec{x}_j}{\sum_i s_i}\right)^2 \\ - \frac{(\sum_j s_j \vec{x}_j)^2}{z \sum_i s_i} + \frac{1}{z}\sum_i s_i x_i^2 \end{aligned} \tag{40}$$

This allows to compute the integral over $\vec{x}$, yielding

$$I = \prod_i \frac{1}{\Gamma(\Delta_i)} \int dz ds_i \prod_i s_i^{\Delta_i - 1} \frac{1}{z^{d+1}} \frac{\pi^{d/2} z^{d/2}}{(\sum_i s_i)^{d/2}} \exp\left(-\sum_i \lambda_i z - \sum_i \frac{s_i \vec{x}_i^2}{z} + \frac{(\sum_j s_j \vec{x}_j)^2}{\sum_i s_i z}\right) \tag{41}$$

Rescaling $z \to z/\sum_i s_i$, this gives

$$I = \pi^{d/2} \prod_i \frac{1}{\Gamma(\Delta_i)} \int_0^\infty dz ds_i \frac{1}{z^{d/2+1}} \exp\left(-z - \frac{s_1 s_2 \vec{x}_{12}^2 + s_1 s_3 \vec{x}_{13}^2 + s_2 s_3 \vec{x}_{23}^2}{z}\right) \tag{42}$$

where $\vec{x}_{ij} \equiv \vec{x}_i - \vec{x}_j$. This integral can be expressed in terms of Γ functions. To see it, we do the change of variables from (z, s_1, s_2, s_3) to (z, t_1, t_2, t_3) defined by

$$s_i = \frac{\sqrt{z t_1 t_2 t_3}}{t_i} \tag{43}$$

The Jacobian can be computed straight-forwardly and reads

$$\left|\frac{\partial(z, s_i)}{\partial(z, t_i)}\right| = \frac{z^{3/2}}{2\sqrt{t_1 t_2 t_3}} \tag{44}$$

The nice property of this change of variables is that

$$\frac{s_1 s_2 \vec{x}_{12}^2 + s_1 s_3 \vec{x}_{13}^2 + s_2 s_3 \vec{x}_{23}^2}{z} = t_3 \vec{x}_{12}^2 + t_2 \vec{x}_{13}^2 + t_1 \vec{x}_{23}^2 \tag{45}$$

Altogether,

$$
\begin{aligned}
I = \frac{\pi^{d/2}}{2} \prod_i \frac{1}{\Gamma(\Delta_i)} \int_0^\infty dz dt_i z^{\frac{\Delta_1+\Delta_2+\Delta_3-d}{2}-1} t_1^{-1+\frac{\Delta_2+\Delta_3-\Delta_1}{2}} t_2^{-1+\frac{\Delta_1+\Delta_3-\Delta_2}{2}} \\
t_3^{-1+\frac{\Delta_1+\Delta_2-\Delta_3}{2}} e^{-z-t_3\vec{x}_{12}^2-t_2\vec{x}_{13}^2-t_1\vec{x}_{23}^2}
\end{aligned}
\tag{46}
$$

Once rescaling $t_3 \to t_3/\vec{x}_{12}^2$, and similarily for t_1 and t_2, we recognize the definition of gamma functions. Thus, plugging back $d = 4$ (AdS_5) we obtain

$$
\begin{aligned}
\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle &= -\frac{\lambda}{\pi^6} \left(\prod_i \frac{\Gamma(\Delta_i)}{\Gamma(\Delta_i - 2)} \right) I \\
&= \frac{\lambda a_1}{|\vec{x}_1 - \vec{x}_2|^{\Delta_1+\Delta_2-\Delta_3} |\vec{x}_1 - \vec{x}_3|^{\Delta_1+\Delta_3-\Delta_2} |\vec{x}_2 - \vec{x}_3|^{\Delta_2+\Delta_3-\Delta_1}}
\end{aligned}
\tag{47}
$$

with

$$
\begin{aligned}
a_1 = -\frac{\Gamma\left[\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)\right] \Gamma\left[\frac{1}{2}(\Delta_1+\Delta_3-\Delta_2)\right] \Gamma\left[\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)\right]}{2\pi^4 \Gamma(\Delta_1-2)\Gamma(\Delta_2-2)\Gamma(\Delta_3-2)} \\
\cdot \Gamma\left[\frac{1}{2}(\Delta_1+\Delta_2+\Delta_3) - 2\right]
\end{aligned}
\tag{48}
$$